# Leader Selection for Performance and Control of Complex Networks

Professor Linda Bushnell
Department of Electrical Engineering
University of Washington, Seattle
Email: lb2@uw.edu

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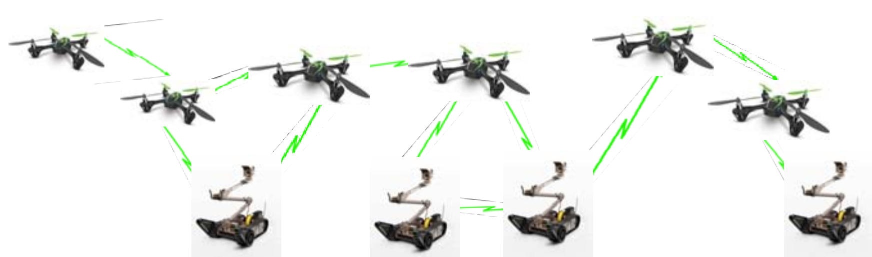
Joint work with Andrew Clark and Radha Poovendran

# Example: Search and Rescue

- Network: Consists of aerial and ground robots; wireless network
- Each node: computes location, trajectory & coverage needed for search and rescue
- Communication and coordination tasks are performed in a distributed, autonomous manner

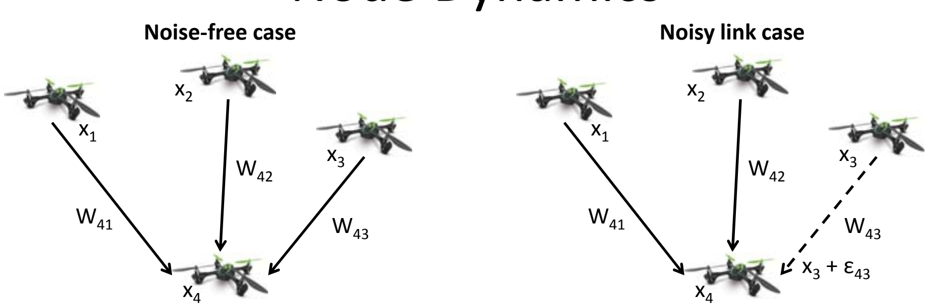


# Features

- Node
  - Mobile
  - Battery resource limited
  - Forms wireless links; Limited communication range; lossy channel between nodes
- Network
  - **Time varying topology** due to mobility and node/link failure
- Formation and Control
  - Needs to be distributed and adaptively computable
  - Control and communication protocols need to be resource efficient



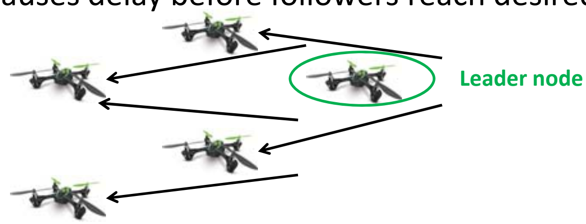
# Node Dynamics




- Each node monitors and controls its own internal state (position, velocity, angular velocity)
- Receives state information as input from neighbors
- Computes and broadcasts internal state using predefined rule
  - Common approach: Weighted averaging of neighbor states
- Link noise affects the estimates and affects state update (errors in position, velocity estimates)

# Leader-Follower System

- In a large network, impractical to provide control inputs to each node
- Instead, small subset of **leader nodes** act as control inputs to influence the follower nodes to a desired state
- Leader inputs propagate through network via local state updates
- Propagation causes delay before followers reach desired states




5

# Examples of Leader-Follower Systems

- Steering formations of unmanned vehicles
- Localization in sensor networks
- Influence propagation in social networks
- Control of gene regulation and expression
- Synchronization of neuronal networks and biological oscillators
- **Main question:** Which nodes are effective leaders? How to efficiently choose leader nodes?

6

# Today's Talk: How to Select Leaders

- Performance:
  – Robustness to link noise
  – Minimum error in intermediate states prior to convergence
- Controllability
  – Ability to drive follower node states to any value within finite time
- Joint performance and controllability

7

# Outline

- Motivating Application
- Leader-Follower System Requirements
- Leader Selection in Complex Networks
  – Robustness to link noise
  – Minimizing convergence error
  – Performance and Controllability
- Conclusions and Future Work

8

# System Model

- Network of n nodes, indexed V = {1,…,n}, edge set E
- Each follower node has state $x_i(t)$ with dynamics
$$\dot{x}_i(t) = -\sum_{j \in N(i)} W_{ij}(x_i - x_j) + \epsilon_{ij}(t)$$
- $\varepsilon_{ij}(t)$ zero-mean white process with variance $v_{ij}$, $v_{ij} = v_{ji}$
- Nodes in the leader set, denoted S, maintain a constant state x*
- Weights $W_{ij}$ given by $W_{ij} = \nu_{ij}^{-1}$
- Dynamics have vector form $\dot{\mathbf{x}}_f(t) = -L\mathbf{x}(t) + \mathbf{w}(t)$, where w is a zero-mean white process and L is defined by

$$L_{ij} = \begin{cases} W_{ij}, & (i,j) \in E \\ \sum_{l \in N(i)} W_{il}, & i = j \\ 0, & i \in S \\ 0, \text{else} & \end{cases}$$

9

# Related Work – Link Noise

- Analysis of mean-square error due to noise in a network with given leaders and dynamics
  - Noise in agent state updates [Patterson & Bamieh `10, Young et al `10]
  - Noise in communication links [Barooah & Hespanha `06]
  - Quantization noise [Kar & Moura `09]
- Leader selection under link noise via convex relaxation [Lin et al `11, Fardad et al `11]
  - Does not provide optimality guarantees
- Currently, no efficient approach with provable bounds on the optimality of the leader set

10

# Quantifying Error Due to Noise

- The Laplacian matrix L can be decomposed as $L = \left(\begin{array}{c|c} L_{ff} & L_{fl} \\ \hline 0 & 0 \end{array}\right),$

  Here, $L_{ff}$ and $L_{fl}$ represent the influence of followers and leaders
- Theorem (Barooah et al `06): The mean-square error in the follower node states in steady-state is equal to

$$\lim_{t\to\infty} \mathbf{E}||\mathbf{x}(t) - x^*\mathbf{1}||_2 = \mathbf{tr}(L_{ff}^{-1})$$

- We define the error due to link noise metric $R(S) = \mathbf{tr}(L_{ff}^{-1})$ as trace of the steady-state covariance matrix of follower nodes
- Define $R(S, u) = (L_{ff}^{-1})_{uu}$ for $u \in V \backslash S$ as the variance of each follower node



# Problem Formulation

- Selecting up to k leaders to minimize error due to noise

$$\begin{array}{ll} \text{minimize} & R(S) \\ \text{s.t.} & |S| \leq k \end{array}$$

- Selecting the minimum-size leader set to achieve a bound α on error due to link noise

$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & R(S) \leq \alpha \end{array}$$

- **Our Approach:** Prove **supermodularity** of R(S) as a function of S

# Supermodularity

- Let V be a finite set; a function $f : 2^V \to \mathbb{R}$ supermodular if for any $S \subseteq T \subseteq V$ and $v \in V \setminus T$,

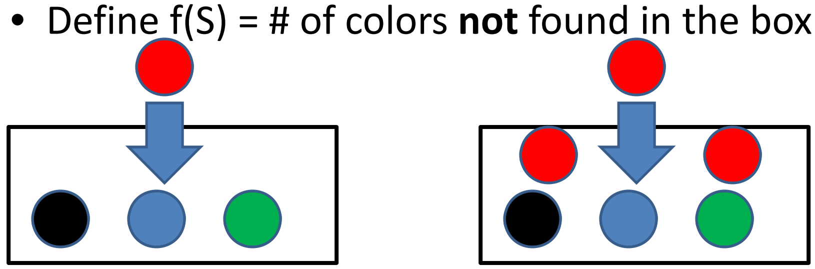
$$f(S) - f(S \cup \{v\}) \geq f(T) - f(T \cup \{v\})$$

- A diminishing returns property for set functions
  - E.g., cost functions
- If f is supermodular, then -f is submodular
- Efficient approximation algorithms for minimizing supermodular functions exist

13

# An Example

- Consider a collection of balls of different colors (e.g., Red, Green Blue)
- A set S of balls is placed in a box
- Define f(S) = # of colors **not** found in the box



**Number of colors not found is reduced by one**

**No effect on number of colors represented**

14

# Proving R(S) is Supermodular

- **First step:** R(S,u) is equal to the graph effective resistance between u and S (set S to 0 volts and node u to one volt; measure the resistance between the set S and node u)
- **Second step:** The effective resistance is proportional to the commute time κ(S,u) of a random walk from u to S
  - Generalization of [Chandra et al `89]
- **Third step:** The commute time κ(S,u) is supermodular as a function of S

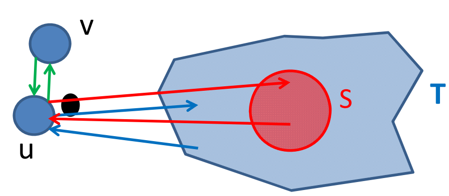
15

# Commute Time is Supermodular

- **Commute time:** Expected time for random walk starting at u to reach any node in S and return to u
- For any $S \subseteq T$, need to show

$$\kappa(S,u) - \kappa(S \cup \{v\}, u) \geq \kappa(T,u) - \kappa(T \cup \{v\}, u)$$

**κ(T,u) = 2 steps**
**κ(T∪{v},u) = 2 steps**
**κ(S∪{v},u) = 4 steps**
**κ(S,u) = 6 steps**




16

# Back to R(S)

- R(S,u), which is proportional to κ(S,u) is supermodular
- $R(S) = \sum_{u \in V \setminus S} R(S,u)$ is supermodular
- The supermodularity property leads to provable guarantees for a greedy algorithm

17

# Choosing up to k Leaders

- Choose set *S* of *k* leaders that minimizes total error

$$\begin{array}{ll} \text{minimize} & R(S) = \sum_{u \in V} R(S,u) \\ \text{s.t.} & |S| \le k \end{array}$$

- <u>Greedy Selection Procedure:</u>
  - Initialize leader set *S* to empty set
  - At each iteration, add the node *v* to *S* that maximizes
    *R(S) – R(S+v) (largest incremental decrease in error)*
  - Stop after *k* iterations
- Guaranteed upper bound: $\left(1 - \frac{1}{e}\right) R^* + \frac{1}{e} R_{max}$
  - *R** is optimum, $R_{max} \triangleq \max_i \sum_{u \in V} R(i,u)$

18

# Choosing Leaders to Achieve Error Bound

- Choose minimum-size set *S* to meet bound *α* on error

$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & \sum_{u \in V} R(S,u) = R(S) \leq \alpha \end{array}$$

- Selection Procedure:
  - Initialize S' = ∅
  - At each iteration, add the node *v* to *S* that maximizes *R(S) – R(S+v)*
  - Stop when *R(S) ≤ α*
- Guaranteed bound: $|S| \leq |S^*| \left(1 + \log\left\{\frac{R_{max}}{R(S^*)}\right\}\right)$
  - *S** is optimum set, $R_{max} \triangleq \max_{i} \sum_{u \in V} R(i,u)$

19

# Choosing Leaders with Switching Between Topologies

- Network may switch between topologies $G_1,\ldots,G_r$
- First problem: Minimize average error

$$\frac{1}{r}\sum_{i=1}^{r} R(S|G_i)$$

  – Nonnegative weighted sum of supermodular functions
- Second problem: Minimize worst-case error

$$\max_{i=1,\ldots,r} R(S|G_i)$$

  – Not a supermodular function! Cannot use techniques above.

20

# Minimizing Worst-case Error

- Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & \max_{i=1,\ldots,r} R(S|G_i) \leq \alpha \end{array}$$

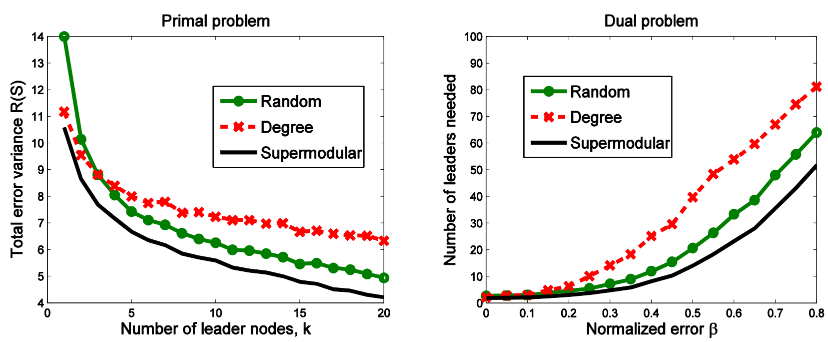
- This is equivalent to

$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & \frac{1}{r}\sum_{i=1}^{r} \max\{R(S|G_i), \alpha\} \leq \alpha \end{array}$$

- Lemma: $F_i(S) \triangleq \max\{R(S|G_i), \alpha\}$ is a supermodular function of S
- Hence, the equivalent optimization problem can be approximated using a greedy algorithm

21

# Numerical Results – Static Case




- Simulated network of 100 randomly positioned nodes
- Edge between two nodes if within communication range
- Supermodular optimization provides lower bound on random and degree-based selection

22

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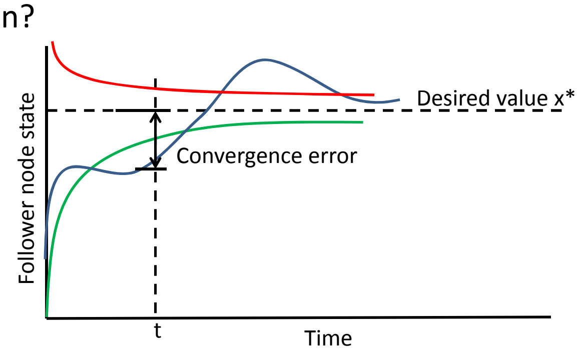
23

# Convergence Error

- **Goals of system:**
  - Ensure that follower nodes converge to a desired state
  - Reduce deviations from desired state prior to convergence
- How to minimize these convergence errors via leader selection?




24

## System Model

- Consider follower node dynamics without noise:

$$\dot{x}_i(t) = -\sum_{j \in N(i)} W_{ij}(x_i - x_j)$$

- Weights $W_{ij}$ are arbitrary and nonnegative
- Each leader node $j \in S$ maintains distinct constant state $x_i^*$
- Vector form $\dot{\mathbf{x}}(t) = -L\mathbf{x}(t)$

25

## Related Work – Convergence Error

- Convergence analysis for given leader set
  - Fixed and switching networks [Jadbabaie et al 2003]
  - Directed and time-delayed networks [Olfati-Saber et al 2004]
  - Stochastic networks [Hatano & Mesbahi, 2005]
  - Spectral bounds on convergence rate [Rahmani et al 2009]
- Link weight selection to minimize convergence error [Boyd 2006]
  - Semidefinite programming approach
  - Does not consider impact of leader nodes
- Currently, no efficient approach for selecting leader nodes to minimize convergence error

26

## Convergence Error Metric and Problem Formulation

- Let $A = \{x_j^* : j \in S\}$, and let $\overline{A}$ denote the convex hull of A
- The **convergence error (or containment error)** at time t is defined by the distance to the convex hull

$$f_t(S) \triangleq \left(\sum_{i \in V \setminus S} (d(x_i(t,S), \overline{A})^p)\right)^{1/p} = \left(\sum_{i \in V \setminus S} \min_{y \in \overline{A}} \{|x_i(t,S) - y|^p\}\right)^{1/p}$$

- Problem of selecting up to k leaders:

$$\begin{array}{ll} \text{minimize} & f_t(S) \\ \text{s.t.} & |S| \leq k \end{array}$$

- Problem of selecting minimum-size leader set:

$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & f_t(S) \leq \alpha \end{array}$$

27

## Our Approach

1. Derive upper bound $\hat{f}_t(S)$ on convergence error at time t that is independent of the initial state **x**(0)
2. Establish a connection between the upper bound and the probability that a random walk on the graph reaches the leader set in time t
3. Prove that the probability of reaching the leader set is supermodular as a function of S
4. Prove that the upper bound on the convergence error is supermodular as a function of S

28

## Selecting up to k Leaders

- Supermodularity implies that a simple greedy algorithm gives a provable bound on the optimal leader set
- To select a set of **up to k leaders** to minimize $\hat{f}_t(S)$:
  - Initialize S' = ∅
  - At each iteration, choose v* that maximizes

$$\hat{f}_t(S') - \hat{f}_t(S' \cup \{v\})$$

  - Set S' = S' ∪ {v*}, terminate when |S'| = k
- **Theorem:** If S* is the optimal set, then

$$\hat{f}_t(S) \leq \left(1 - \frac{1}{e}\right) \hat{f}_t(S^*) + \frac{1}{e} f_{max}$$

where $f_{max} \triangleq \max\{\hat{f}_t(\{v\}) : v \in V\}$

(Follows from Nemhauser et al `78)

29

## Selecting Leaders to Achieve Error Bound

- In order to achieve a bound α on $\hat{f}_t(S)$ :
  - Initialize S' = ∅
  - At each iteration select v* that maximizes

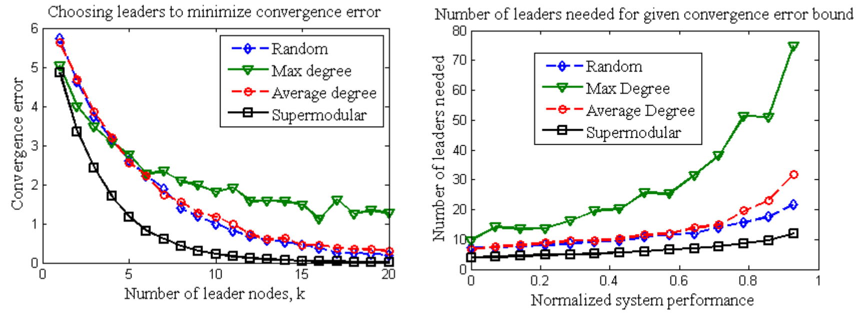
$$\hat{f}_t(S') - \hat{f}_t(S' \cup \{v\})$$

  - Set S' = S' ∪ {v*}, terminate when $\hat{f}(S') \leq \alpha$
- **Theorem:** Let S* be the minimum-size set with $\hat{f}_t(S) \leq \alpha$
  - We have:

$$\frac{|S'|}{|S^*|} \leq 1 + \ln\left(\frac{f_{max}}{\alpha}\right)$$

30

# Simulation Results




- Consider an undirected graph with 100 nodes
- Two nodes share link if within communication range
- Supermodular optimization provides lowest convergence error and requires fewest leaders

31

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32

# Controllability

- A system with dynamics

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

is controllable if for any states **a**, **b** with **x**(0) = **a** and any T > 0, there exists {u(t) : 0≤ t ≤ T} such that **x**(T) = **b.**

- Equivalently, it is possible to drive the state **x** from any initial state to any final state in finite time

33

# Related Work

- Controllability analysis for a given leader set
  - Necessary and sufficient graph spectrum conditions for controllability [Tanner 2004]
  - Necessary graph-based conditions for controllability [Rahmani et al 2009]
  - Controllability of dynamic networks [Liu et al 2008]
- Efficient algorithm for leader selection for controllability [Liu et al 2011]
  - No performance guarantees
- Currently no technique for joint consideration of performance and controllability

34

# Joint Performance and Controllability

- **Goal of this topic:** Joint optimization of controllability and performance criteria
- **Approach:** Introduce a graph controllability index (GCI)
  - Characterizes the largest controllable subgraph of the network
  - Prove submodularity of the GCI
  - Formulate joint performance and controllability as a submodular optimization problem

35

# Graph Controllability Index

- Define **GCI** as the largest controllable subgraph:

$$c(S) \triangleq \max\{|V'| : (V', E') \subseteq G \text{ is controllable from } S\}$$

- The controllability can then be traded off with a performance metric f(S) via the optimization problem

$$\begin{array}{ll} \text{maximize} & \frac{1}{n}c(S) - \lambda f(S) \\ \text{s.t.} & |S| \leq k \end{array}$$

- Possible objective functions f(S):
  - Mean-square error due to link noise
  - Convergence error
- Computation of GCI is based on structural controllability of the graph

36

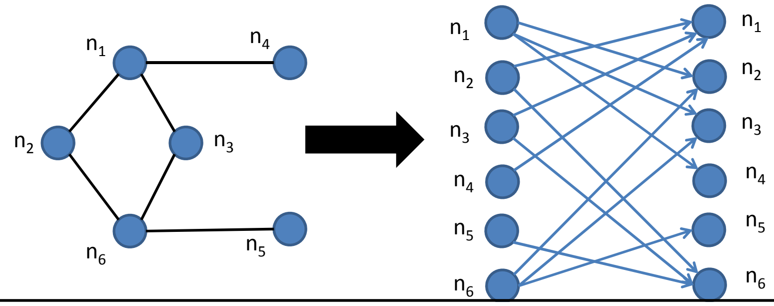
# Structural Controllability

- Consider a system with state $\mathbf{x}(t) \in \mathbf{R}^m$, input $\mathbf{u}(t) \in \mathbf{R}^l$, and dynamics $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$
- *Structural controllability [Lin `74]* holds if, for almost every choice of the nonzero entries of (A,B), system is controllable
- Define a graph G with vertex set $\{v_1,\dots,v_m,w_1,\dots,w_l\}$ by adding edge $(v_j, v_i)$ if $A_{ij} \neq 0$ and edge $(w_j, v_i)$ if $B_{ij} \neq 0$
  - Here, $A = L_{ff}$ and $B = L_{fl}$
- Theorem (Lin `74): (A,B) satisfies structural controllability iff:
  1. For each $v_i$, there exists $w_j$ such that a path exists from $w_j$ to $v_i$
  2. For each $T \subseteq \{v_1,\dots, v_m\}$, $|T| \leq |N(T)|$, where N(T) is set of neighbors of T

37

# Controllability and Matching

- If graph is connected, then SC holds iff, for any $A \subseteq V \backslash S$, $|A| \leq |N(A)|$
- Consider the bipartite representation of G
- By Hall Marriage Theorem [Brualdi `10], SC is equivalent to existence of a **perfect matching** from N(V\S) into V\S.
- We prove submodularity of the GCI using this connection to graph matching




38

# Algorithms for Maximizing GCI

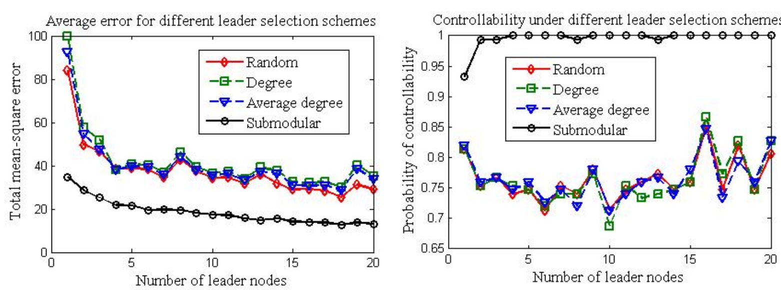
- A greedy approach maximizes GCI up to provable bound
- At each iteration, select the agent v such that

$$\frac{1}{n}(c(S+v)-c(S))-\lambda(f(S+v)-f(S))$$

  is maximized
- Special case: $\lambda$=0
  - Reduces to optimization over controllability only
  - If k is sufficiently large, then algorithm returns the minimum-size leader set needed for SC in polynomial time
  - Reduces to a graph matching, resulting in efficient leader selection with identical guarantees as existing methods

39

# Simulation Results




- Erdos-Renyi random graph G(n,p), with n=100 and p=0.05 simulated.
- Simulations compare submodular, random, and degree-based algorithms using network coherence as performance metric
- Submodular approach outperforms random and degree-based
- Achieves controllability when possible
- Degree-based selection provides controllability in roughly three-fourths of cases

40

# Conclusions

- Presented a unifying, supermodular optimization framework for leader selection based on:
  - Robustness to link noise
  - Smooth convergence to desired state
  - Controllability & performance
- Developed efficient algorithms with provable guarantees in static and dynamic networks
- Based on connections between networked dynamical systems and the theory of random walks on graphs



# Future Work

- Distributed algorithms for leader selection
- Controllability of dynamic networks
- Different application domains:
  - Biological networks
  - Social networks
  - Unmanned vehicular networks
- Leader selection for security as well as control

# References


1. A. Clark, L. Bushnell, and R. Poovendran, "A Supermodular Optimization Framework for Leader Selection under Link Noise in Linear Multi-Agent Systems," arXiv preprint arXiv:1208.0946, 2012.
2. A. Clark, L. Bushnell, and R. Poovendran, "Leader Selection for Minimizing Convergence Error in Leader-Follower Systems: A Supermodular Optimization Approach," 10th International Symposium on Optimization in Mobile, Wireless, and Ad Hoc Networks (WiOpt), pp. 111-15, 2012.
3. A. Clark, L. Bushnell, and R. Poovendran, "On Leader Selection for Performance and Controllability in Multi-Agent Systems," 51st IEEE Conference on Decision and Control, pp. 86-93, 2012.

Web link 1: http://www.ee.washington.edu/research/ncs/research.html
Web link 2: http://www.ee.washington.edu/research/nsl/faculty/radha/


43

# Questions?

- Thank you for your time and attention

44