On the stabilization of positive switched systems: state of the art and open problems

Maria Elena Valcher

University of Padova, Italy

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What is a switched system?

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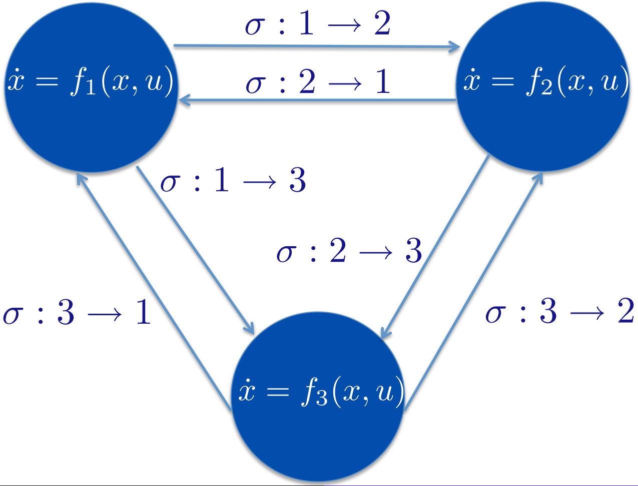
- One can think of each subsystem as describing the system evolution in a specific working mode.
- The switching among the subsystems can be considered either as an external signal that we cannot control, or as a control input we have on the system.

What is a switched system?



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- If the variables involved in the system description are constrained by their physical nature to take positive values (e.g. they may represent concentrations, probabilities, population levels, temperatures, etc.) then we have positive switched systems.
- So, positive switched systems consist of a family of “positive" subsystems and a rule to switch among them.

Example: HIV viral mutation problem

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- ART drugs reduce the growth of certain viral populations, while leaving that of others unchanged.
- One of the main problems in HIV infection is that resistant mutations have been described for all antiretroviral drugs currently in use.
- This led to the conclusion that switching therapeutic options will be required lifelong in order to prevent HIV disease progression.

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- p alternative ART therapies are available: the therapy used at time t is indicated by $\sigma(t)$ and takes values in the finite set $\{1, 2, \ldots, p\}$.

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This yields the following system

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$$\dot{x}(t) = A_{\sigma(t)} x(t),$$

where $A_{\sigma(t)} := R_{\sigma(t)} - \delta I_n + \mu M$ is a matrix with positive off-diagonal entries (Metzler matrix) for every $\sigma(t) \in \{1, 2 \dots, p\}$.

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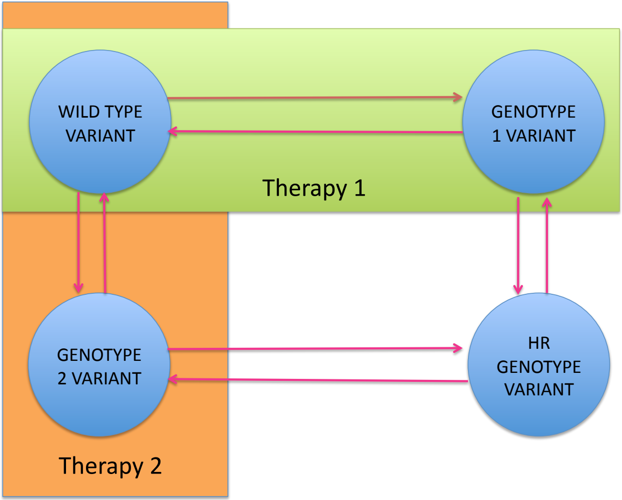

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with $A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1/4 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 1 \\ 1/4 & -1 \end{bmatrix}$ is a CPSS.

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$$\|x(t)\| < Ce^{-\beta t}\|x(0)\|, \qquad \forall t \geq 0;$$

- consistently exponentially stabilizable if there exists a switching signal $\bar{\sigma}(t), t \in \mathbb{R}_+,$ such that the resulting (time-varying) system is exponentially stable.

Stabilizability properties: equivalent conditions

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(iii) the system is consistently exponentially stabilizable by means of a periodic switching sequence $\bar{\sigma}(t), t \in \mathbb{R}_+$.

Stabilization problem: Lyapunov based strategies (1)

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As for non-positive switched systems, the idea is to resort to control Lyapunov functions.

By exploiting the system positivity, we may look for Lyapunov functions $V(x)$ that are co-positive, by this meaning that they take positive values for positive values of x.

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$$V(x(t+h)) \leq V(x(t)) - \int_t^{t+h} \phi(x(\tau))d\tau, \qquad \forall\ h, t \geq 0,$$

holds for every state trajectory $x(t), t \in \mathbb{R}_+$, generated according to σ.

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(i) the system is exponentially stabilizable;

(ii) the system admits a positively homogeneous, concave and co–positive control Lyapunov function.

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(iv) The CPSS admits a quadratic positive definite control Lyapunov function $V_{PD}(x) = x'Px$, with $P = P' \succ 0$.

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Example 2 Consider the three-dimensional CPSS, switching among three subsystems characterized by the matrices:

$$A_1 = \begin{bmatrix} 0.01 & 0 & 1 \\ 0 & -0.99 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.01 & 0 & 0 \\ 1 & 0.01 & 0 \\ 0 & 0 & -0.99 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.99 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 1 & 0.01 \end{bmatrix}.$$

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It is easy to prove that no convex Hurwitz combination of the three matrices can be found.

Lyapunov-based sufficient conditions (4)

However, the matrix product $e^{A_1}e^{A_2}e^{A_3}$ is Schur. So, the periodic switching law

$$\sigma(t) = \begin{cases} 3, & t \in [3k, 3k+1); \\ 2, & t \in [3k+1, 3k+2); \\ 1, & t \in [3k+2, 3k+3); \end{cases} \qquad k \in \mathbb{Z}_+,$$

makes the resulting system (consistently) exponentially stable.

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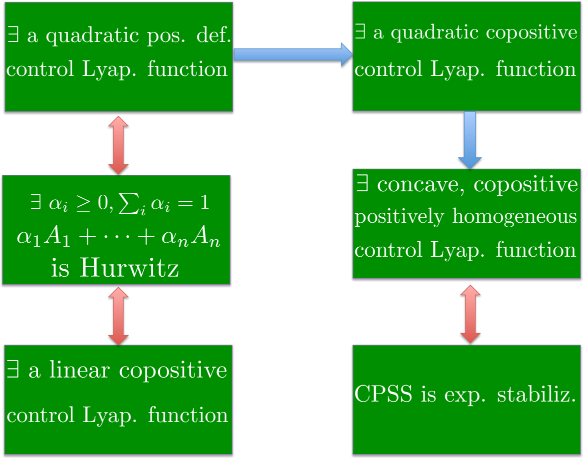
However, the existence of a quadratic co-positive control Lyapunov function of arbitrary rank is a weaker condition, and it ensures, in turn, exponential stabilizability.

Lyapunov functions: scheme

So, the situation for CPSS is



Stabilizability: the case of two-dimensional systems

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Theorem 2 For a two-dimensional CPSS the following facts are equivalent:

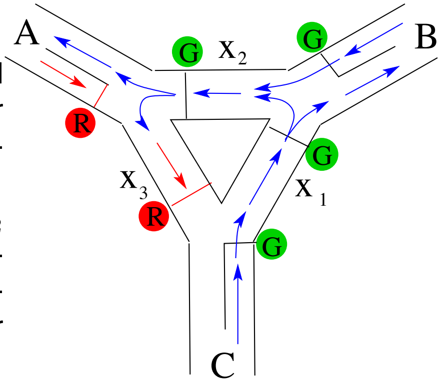
(i) the CPSS is exponentially stabilizable;

(ii) there exist indices $i_1, i_2 \in \mathcal{P}$ and numbers $\alpha_1, \alpha_2 \geq 0$, with $\alpha_1 + \alpha_2 = 1$, such that $\alpha_1 A_{i_1} + \alpha_2 A_{i_2}$ is Hurwitz.

A simple traffic congestion example (1)

Three main roads (A, B and C) converge into a “triangular connection" governed by traffic lights.

The buffer variables, x_1, x_2 and x_3, represent the vehicles waiting at the three traffic lights inside the triangular loop.



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The other two configurations are obtained by a circular rotation of x_1, x_2 and x_3 (as well as of A, B and C).

A simple traffic congestion example (3)

So, we get a CPSS with subsystem matrices

$$A_1 = \begin{bmatrix} 0 & 0 & \beta \\ 0 & -\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix} \; A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & -\gamma \end{bmatrix} \; A_3 = \begin{bmatrix} -\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix},$$

with $\gamma, \beta > 0$. The matrix A_i corresponds to the situation when the i-th traffic light is red.

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with $\gamma, \beta > 0$. The matrix A_i corresponds to the situation when the i-th traffic light is red.
If $\beta < \gamma$, all convex combinations of A_1, A_2 and A_3 are Hurwitz.
So, there is always exponential stabilizability.

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Indeed, switching the red light from 3 to 1 allows for a "fast recovery" of the congestion on x_3, while switching from 3 to 1 leaves such congestion unchanged.

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Thanks for your attention!